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Determining Sets and Korovkin Sets on the Circle

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In this paper we investigate properties of a subspace X spanned by a Chebyshev system on the circle. In particular we show that the set of operators for which X is a positive operator Korovkin set is equivalent to the set of operators for which X is a determining set. These results are obtained by applying uniqueness properties of the moment problem.

Let $C = C(T)$ be the Banach space of real-valued continuous functions on the circle T with the uniform norm. We denote by $\mathscr{B}(C)$ the space of all bounded linear operators on C. Let C^* be the space of all bounded linear functionals on C. For an operator S in $\mathscr{B}(C)$ let S^* be the dual operator defined on C^*. For a point p in T, let $\hat{p}$ denote the functional in C^* given by evaluation at the point p. Suppose g is a function on a set A containing the set B. Then $g|_B$ is the restriction of g to B.

We define $\mathscr{T}_+$ to be the cone of positive linear operators in $\mathscr{B}(C)$; i.e., $S \in \mathscr{T}_+$ if $f \geqslant 0$ implies $Sf \geqslant 0$. We say that a subspace X of C is a $\mathscr{T}_+$-*determining set* for an operator S in $\mathscr{T}_+$ if for any R in $\mathscr{T}_+$ the equality $Rf = Sf$ for all f in X implies $R = S$. This concept was introduced by Shashkin [10]. We say that X is a $\mathscr{T}_+$-*Korovkin set* for an operator S in $\mathscr{T}_+$ if for any sequence $\{S_n\}$ in $\mathscr{T}_+$ the convergence of $S_n f$ to Sf in the uniform norm for all f in X implies the convergence of $S_n f$ to Sf for all f in C. Korovkin sets of this type have been investigated by Micchelli [7, 8], Cavaretta [1], and the author [9]. Let $\mathscr{L}_+$ be the cone of positive functionals in C^*; i.e., $\mu \in \mathscr{L}_+$ if $f \geqslant 0$ implies $\mu(f) \geqslant 0$. The corresponding concepts of an $\mathscr{L}_+$-determining set and an $\mathscr{L}_+$-Korovkin set for positive functionals are defined in the obvious way.

Let X in C be the linear span of a fixed but arbitrary $(2m + 1)$-dimensional Chebyshev system $\{u_0, u_1, ..., u_{2m}\}$. Let K be the positive cone in X^*. Define

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$M = \{\hat{p}\,|_X : p \in T\}$. Clearly, $M \subseteq K$. In fact, for each τ in K there exists an integer $1 \leqslant n \leqslant 2m + 1$ such that

$$\tau = \sum_{i=1}^{n} \alpha_i \hat{p}_i \Big|_X \qquad \text{where} \quad \alpha_i \geqslant 0 \quad \text{and} \quad p_i \in T \quad (1 \leqslant i \leqslant n). \tag{1}$$

We denote the smallest possible n for which (1) holds by $I(\tau)$. If μ is a functional in $\mathscr{L}_+$ with exactly n points in its support or carrier, then we denote n by $I(\mu)$.

The following is our main result.

THEOREM 1. *Let R be an operator in $\mathscr{T}_+$. The following are equivalent:*

(i) *X is a $\mathscr{T}_+$-Korovkin set for R,*

(ii) *X is a $\mathscr{T}_+$-determining set for R,*

(iii) *for each p in T, we have $I(R^*\hat{p}) \leqslant m$.*

First we note that the implication (i) $\to$ (ii) follows easily from the observation that if $S\,|_X = R\,|_X$ for some operator S in $\mathscr{T}_+$ then a sequence $\{S_n\}$ in $\mathscr{T}_+$ is constructed by defining $S_n = S$ for all $n \geqslant 1$. The implications (ii) $\to$ (iii) and (iii) $\to$ (i) will follow directly from Lemma 6 and Lemma 5, respectively.

Theorem 1 also holds when the circle T is replaced by a closed interval $[a, b]$ of the real line. This result is due to the author [9]. However, the present setting appears to be more appropriate because a more concise theorem and proof are possible.

We use the following two results. The first characterizes Korovkin sets (see Micchelli [7] or Ferguson [3]). The second (see Karlin and Studden [5, p. 181]) gives conditions under which the moment problem has a unique solution.

THEOREM 2. *For a functional μ in $\mathscr{L}_+$, a subspace X is an $\mathscr{L}_+$-Korovkin set for μ if and only if X is a $\mathscr{L}_+$-determining set for μ. For an operator R in $\mathscr{T}_+$, a subspace X is a $\mathscr{T}_+$-Korovkin set for R if and only if X is a $\mathscr{T}_+$-determining set for $R^*\hat{p}$ for each p in T.*

THEOREM 3. *A functional τ in K is a boundary point of K if and only if $I(\tau) \leqslant m$. Furthermore, each boundary point admits only one representation* (1). *Similarly, for each interior point of K and any t in T there exists a unique representation* (1) *such that $n = m + 1$ and t is in the support.*

The following two lemmas parallel Micchelli's results [7] for continuous functions on a compact interval of the real line.

LEMMA 4. *The subspace X is an $\mathscr{L}_+$-Korovkin set for μ in $\mathscr{L}_+$ if and only if $I(\mu) \leqslant m$.*

Proof. Suppose X is an $\mathcal{L}_+$-Korovkin set for μ. By Theorem 2, X is an $\mathcal{L}_+$-determining set for μ. If $\mu|_X$ is an interior point of K, then for each t in T there exists a representation (1) with t in the support. However, these representations would not all have the same extension to C. This is a contradiction. Therefore $\mu|_X$ is a boundary point of K. By Theorem 3, $I(\mu) \leq m$.

Conversely, suppose $I(\mu) \leq m$. By Theorem 3, $\mu|_X$ is a boundary point of K and $\mu|_X$ admits a unique representation (1). Suppose there exists τ in $\mathcal{L}_+$ such that $\tau|_X = \mu|_X$, but $\tau(g) \neq \mu(g)$ for some g in C. There exists (see Holmes [4, p. 84]) $\alpha_i \geq 0$ and p_i in T, $1 \leq i \leq m+1$ such that for

$$\sigma = \sum_{i=1}^{m+1} \alpha_i \hat{p}_i \tag{2}$$

we have $\sigma|_X = \tau|_X = \mu|_X$, but $\sigma(g) = \tau(g) \neq \mu(g)$. This contradicts the uniqueness of representation (1). Therefore, X is a $\mathcal{L}_+$-determining set for μ. By Theorem 2, X is a $\mathcal{L}_+$-Korovkin set for μ. The lemma is proved.

LEMMA 5. *The subspace X is a $\mathcal{T}_+$-Korovkin set for an operator R in $\mathcal{T}_+$ if and only if for each p in T we have $I(R^*\hat{p}) \leq m$.*

Proof. This result is a consequence of Theorem 2 and Lemma 4.

Shashkin [10] investigated the $\mathcal{T}_+$-determining sets for an operator from $C(Q)$ to $B(Q)$ where Q is a compact metric space and $B(Q)$ is the space of bounded real-valued functions on Q with the supremum norm. He has shown: If a positive operator S has an n-dimensional $\mathcal{T}_+$-determining set, then the support of the functional $S^*\hat{p}$ for each p in Q contains at most $n+1$ points. The proof of this result depends upon the fact that the range space is $B(Q)$ and does not extend to the present case. Using different techniques we obtain a similar result in the more natural situation where the operator has the same domain and range.

LEMMA 6. *If X is a $\mathcal{T}_+$-determining set for an operator R in $\mathcal{T}_+$, then for each p in T we have $I(R^*\hat{p}) \leq m$.*

Proof. Suppose X is a $\mathcal{T}_+$-determining set for R in $\mathcal{T}_+$. First, we show how to define an operator S (depending on t in T) by defining $S^*\hat{p}$ for every p in T. The operator S will satisfy $I(S^*\hat{p}) \leq m+1$ and agree with R on X and, therefore, on C. We then show $I(R^*\hat{p} \leq m)$.

Let t be a fixed, but arbitrary element of T. For each p in T we select a representation denoted by $\tau(p)$ for $(R^*\hat{p})|_X$ of the form (1). Given such a representation there exists the natural extension denoted by $\sigma(p)$ to all of C of the form (2). The choice of representation (1) depends on the location of $(R^*\hat{p})|_X$ as follows: If $(R^*\hat{p})|_X$ is in the boundary of K, then by Theorem 3 there exists a unique representation $\tau(p)$ in X^* of the form (1) such that for the extension $\sigma(p)$ we have $I(\sigma(p)) \leqslant m$. If $(R^*\hat{p})|_X$ is not a boundary point of K, then there exists a unique representation $\tau(p)$ of the form (1) with t in the support and $I(\sigma(p)) = m + 1$. Let S be the operator on C defined by $(Sf)(p) = (\sigma(p))(f)$.

According to the standard representation theorem (see Dunford and Schwartz [2, Theorem VI.7.1]), S is an operator in $\mathscr{B}(C)$ if

(i) $\sup\{\|\sigma(p)\| : p \in T\}$ is finite;

(ii) the map $p \mapsto \sigma(p)$ is continuous with the weak* topology of C^*.

Since X is spanned by a Chebyshev system there exists a function g in X such that $g \geqslant 1$. Then

$$\begin{aligned} \|\sigma(p)\| &= (\sigma(p))(1) \leqslant (\sigma(p))(g) \\ &= (\tau(p))(g) = (R(g))(p) \\ &\leqslant \|Rg\| = c. \end{aligned} \tag{3}$$

Therefore (i) holds. This last equality defines the constant c.

We claim $\sigma(\cdot)$ is a continuous function into C^* with the weak* topology: i.e., if $p_n \to p$ in T, then $(\sigma(p_n))(f) \to (\sigma(p))(f)$ for all f in C. If $\tau(p)$ is in the boundary of K, then, since $I(\sigma(p)) \leqslant m$ and $(\sigma(p_n))(f) \to (\sigma(p))(f)$ for all f in X, we have, by Lemma 4 that $(\sigma(p_n))(f) \to (\sigma(p))(f)$ for all f in C.

Suppose $\tau(p)$ is not in the boundary of K. Let $cB_C{}^*$ be the closed ball in C^* centered at 0 of radius c. By (3), $cB_C{}^*$ contains $\{\sigma(p_n)\}$. By the Banach–Alaoglu theorem, $cB_C{}^*$ is weak* compact. Hence, it is sufficient to show that the only weak* cluster point of $\{\sigma(p_n)\}$ is $\sigma(p)$. Let μ be any cluster point of $\{\sigma(p_n)\}$. Since C is separable $cB_C{}^*$ is metrizable. Thus, there exists a subsequence $\{\sigma(p_{n_j})\}$ such that $\sigma(p_{n_j}) \to \mu$ in the weak* topology of C^*. It is easily seen that $I(\mu) \leqslant m + 1$. Since

$$\sigma(p_{n_j})|_X = (R^*\hat{p}_{n_j})|_X \to (R^*\hat{p})|_X = \sigma(p)|_X$$

we have $\mu|_X = \sigma(p)|_X$.

We wish to show that t is in the support of μ. We assume not and arrive at a contradiction. Let

$$\sigma(p_{n_j}) = \rho(p_{n_j}) + \beta_{n_j}\hat{t}$$

where $\beta_{n_j} > 0$ and such that t is not in the support of $\rho(p_{n_j})$. This is possible since $I(\sigma(p_n)) \leqslant m + 1$. From the construction and by Theorem 3, it is easily seen that $I(\rho(p_{n_j})) \leqslant m$.

By Urysohn's lemma there exists a nonnegative function g in C such that $g(t) = 1$ and $\mu(g) = 0$. Thus,

$$\begin{aligned} (\sigma(p_{n_j}))(g) &= (\rho(p_{n_j}))(g) + \beta_{n_j} \\ &\to \mu(g) = 0. \end{aligned}$$

Since $(\rho(p_{n_j}))(g) \geq 0$ we have $\beta_{n_j} \to 0$. As $\rho(p_{n_j}) \to \mu$ in the weak* topology, we get $I(\mu) \leq m$. By Theorem 3, $\mu|_X = \sigma(p)|_X = \tau(p)$ is a boundary point of K which contradicts our original assumption. Hence, t is in the support of μ.

Since t is in the support of μ and $I(\mu) \leq m + 1$, by the uniqueness of the representation $\tau(p)$ we have $\sigma(p) = \mu$. Since $\sigma(p)$ is the only limit point of $\{\sigma(p_n)\}$, (ii) holds. By the representation theorem S is an operator in $\mathscr{T}_+$.

Since X is a $\mathscr{T}_+$-determining set for R and since $S|_X = R|_X$ we have $S = R$. Consequently, $I(R^*\hat{p}) = I(S^*\hat{p}) = I(\sigma(p)) \leq m + 1$ for all p in T.

Suppose there exists p in T such that $I(R^*\hat{p}) = m + 1$, i.e., $(R^*\hat{p})|_X$ is in the interior of K. Let t be a point in T such that t is not in the support of $R^*\hat{p}$. Since t was arbitrary in the above argument, there exists S in $\mathscr{T}_+$ such that t is in the support of $S^*\hat{p}$ and $S = R$. Therefore, t is in the support of $R^*\hat{p}$. This is a contradiction. The lemma is proved.

If X is not spanned by a Chebyshev system, Theorem 1 is not true as the following example shows. Let $Q = [-\pi, \pi]$ and let X be the linear span of $\{1, \sin, \cos\}$. We claim that X is a $\mathscr{T}_+$-determining set for the identity operator I on $C = C(Q)$.

Suppose for some positive operator S in $\mathscr{B}(C)$ that $S|_X = I|_X$. For any p in $(-\pi, \pi)$ there exists a function g in X such that $g(p) = 0$, but $g(q) > 0$ for q in Q, $q \neq p$. Namely, $g(t) = 1 - \cos p \cos t - \sin p \sin t$. Since $(Sg)(p) = (Ig)(p) = 0$ and since $S^*\hat{p}$ is a positive functional, the support of $S^*\hat{p}$ is contained in the set $\{p\}$. However, $(S1)(p) = (I1)(p) = 1$. Hence, $S^*\hat{p} = \hat{p}$ for all p in $(-\pi, \pi)$. Since S is in $\mathscr{B}(C)$ we have $S^*\hat{p} = \hat{p}$ for all p in $[-\pi, \pi]$, i.e., $S = I$. The claim is true. However, since $\hat{\pi}|_X = (-\pi)^\wedge|_X$ and by Theorem 2, X is not a $\mathscr{T}_+$-Korovkin set for I. By the same argument we have the following corollary.

COROLLARY 7. *Let Q be a compact Hausdorff space, let R be a positive operator in $\mathscr{B}(C)$ where $C = C(Q)$, and let X be a subspace of C. If there exists a dense set P in Q such that for p in P we have that X is an $\mathscr{L}_+$-determining set for $R^*\hat{p}$, then X is a $\mathscr{T}_+$-determining set for R.*

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